

SOME PLETHYSTIC IDENTITIES AND KOSTKA-FOULKES POLYNOMIALS.

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1. INTRODUCTION.

Symmetric functions $\{E_{n,k}(X)\}_{k=1}^n$, defined by the Newton interpolation

$$e_n[X \frac{1-z}{1-q}] = \sum_{k=1}^n (z; q)_k \frac{E_{n,k}(X)}{(q; q)_k}$$

plays an important role in the Garsia-Haglund proof of the q, t -Catalan conjecture, [2].

Let $\Lambda_{\mathbb{Q}(q,t)}^n$ be the space of symmetric functions of degree n , over the field of rational functions $\mathbb{Q}(q, t)$, and let $\nabla : \Lambda_{\mathbb{Q}(q,t)}^n \rightarrow \Lambda_{\mathbb{Q}(q,t)}^n$ be the Garsia-Bergeron operator.

By studying recursions, Garsia and Haglund show that the coefficient of the elementary symmetric function $e_n(X)$ in the image $\nabla(E_{n,k}(X))$ of $E_{n,k}(X)$ is equal to the following combinatorial summation

$$(1.1) \quad \langle \nabla(E_{n,k}(X)), e_n(X) \rangle = \sum_{\pi \in D_{n,k}} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)},$$

where $D_{n,k}$ is the set of all Dyck paths with initial k North steps followed by an East step. Here $\text{area}(\pi)$ and $\text{bounce}(\pi)$ are two numbers associated with a Dyck path π . It is conjectured in [4], more generally, that the $\nabla E_{n,k}(X)$ are ‘‘Schur positive.’’

In [1], using (1.1), Can and Loehr prove the q, t -Square conjecture of the Loehr and Warrington [7].

The aim of this article is to understand the functions $\{E_{n,k}(X)\}_{k=1}^n$ better. We prove that the vector subspace generated by the set $\{E_{n,k}(X)\}_{k=1}^n$ of the space $\Lambda_{\mathbb{Q}(q)}^n$ of degree n symmetric functions over the field

$\mathbb{Q}(q)$, is equal to the subspace generated by

$$\{s_{(k, 1^{n-k})}[X/(1-q)]\}_{k=1}^n,$$

Schur functions of hook shape, plethystically evaluated at $X/(1-q)$.

In particular, we determine explicitly the transition matrix and its inverse from $\{E_{n,k}(X)\}_{k=1}^n$ to $\{s_{(k, 1^{n-k})}[X/(1-q)]\}_{k=1}^n$. The entries of the matrix turns out to be cocharge Kostka-Foulkes polynomials.

We find the expansion of $E_{n,k}(X)$ into the Hall-Littlewood basis, and as a corollary we recover a closed formula for the cocharge Kostka-Foulkes polynomials $\tilde{K}_{\lambda,\mu}(q)$ when λ is a hook shape;

$$\tilde{K}_{(n-k, 1^k)\mu}(q) = (-1)^k \sum_{i=0}^k (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix}.$$

Here, μ is a partition of n whose first column is of height r .

2. BACKGROUND.

2.0.1. Notation. A partition μ of $n \in \mathbb{Z}_{>0}$, denoted $\mu \vdash n$, is a nonincreasing sequence $\mu_1 \geq \mu_2 \geq \dots \geq \mu_k > 0$ of numbers such that $\sum \mu_i = n$. The conjugate partition $\mu' = \mu'_1 \geq \dots \geq \mu'_s > 0$ is defined by setting $\mu'_i = |\{\mu_r : \mu_r \geq i\}|$.

$Par(n, r)$ denotes the set of all partitions $\mu \vdash n$ whose biggest part is equal to $\mu_1 = r$.

We identify a partition μ with its Ferrers diagram, in French notation. Thus, if the parts of μ are $\mu_1 \geq \mu_2 \geq \dots \geq \mu_k > 0$, then the corresponding Ferrers diagram have μ_i lattice cells in the i^{th} row (counting from bottom to up).

Following Macdonald, [8] the arm, leg, coarm and coleg of a lattice square s are the parameters $a_\mu(s)$, $l_\mu(s)$, $a'_\mu(s)$ and $l'_\mu(s)$ giving the number of cells of μ that are respectively strictly EAST, NORTH, WEST and SOUTH of s in μ .

Given a partition $\mu = (\mu_1, \mu_2, \dots, \mu_k)$, we set

$$(2.1) \quad n(\mu) = \sum_{i=1}^k (i-1)\mu_i = \sum_{s \in \mu} l_\mu(s).$$

We also set

$$(2.2) \quad \tilde{h}_\mu(q, t) = \prod_{s \in \mu} (q^{a_\mu(s)} - t^{l_\mu(s)+1}) \quad \text{and} \quad \tilde{h}'_\mu(q, t) = \prod_{s \in \mu} (t^{l_\mu(s)} - q^{a_\mu(s)+1}).$$

Let \mathbb{F} be a field, and let $X = \{x_1, x_2, \dots\}$ be an alphabet (a set of indeterminates). The algebra of symmetric functions over \mathbb{F} with the variable set X is denoted by $\Lambda_{\mathbb{F}}(X)$.

If $\mathbb{Q} \subseteq \mathbb{F}$, it is well known that $\Lambda_{\mathbb{F}}(X)$ is freely generated by the set of power-sum symmetric functions

$$\{p_r(X) : r = 1, 2, \dots \text{ and } p_r(X) = x_1^r + x_2^r + \dots\}.$$

The algebra, $\Lambda_{\mathbb{F}}(X)$ has a natural grading (by degree).

$$\Lambda_{\mathbb{F}}(X) = \bigoplus_{n \geq 0} \Lambda_{\mathbb{F}}^n(X),$$

where $\Lambda_{\mathbb{F}}^n(X)$ is the space of homogenous symmetric functions of degree n .

A basis for the vector space $\Lambda_{\mathbb{F}}^n(X)$ is give by the set $\{p_\mu\}_{\mu \vdash n}$,

$$(2.3) \quad p_\mu(X) = \prod_{i=1}^k p_{\mu_i}(X), \text{ where } \mu = \sum_{i=1}^k \mu_i.$$

Another basis for $\Lambda_{\mathbb{F}}^n(X)$ is given by the Schur functions $\{s_\mu(X)\}_{\mu \vdash n}$, where $s_\mu(X)$ is defined as follows. Let

$$(2.4) \quad e_n(X) = \sum_{1 \leq i_1 < \dots < i_n} x_{i_1} x_{i_2} \dots x_{i_n}$$

be the n 'th elementary symmetric function. If $\mu = \sum_{i=1}^k \mu_i$, then

$$(2.5) \quad s_\mu(X) = \det(e_{\mu'_i - i + j}(X))_{1 \leq i, j \leq m},$$

where μ'_i is the i 'th part of the conjugate partition $\mu' = (\mu'_1, \dots, \mu'_l)$ and $m \geq l$.

2.0.2. Plethysm. For the purposes of this section, we represent an alphabet $X = \{x_1, x_2, \dots\}$ as a formal sum $X = \sum x_i$. Thus, if $Y = \sum y_i$ is another alphabet, then

$$(2.6) \quad XY = \left(\sum x_i\right)\left(\sum y_i\right) = \sum_{i,j} x_i y_j = \{x_i y_j\}_{i,j \geq 1},$$

and

$$(2.7) \quad X + Y = \left(\sum_i x_i \right) + \left(\sum_j y_j \right) = \{x_i, y_j\}_{i,j \geq 1}.$$

The formal additive inverse, denoted $-X$, of an alphabet $X = \sum x_i$ is defined so that $-X + X = 0$.

In this vein, if $p_k(X) = \sum_{i \geq 1} x_i^k$ is a power sum symmetric function, we define

$$(2.8) \quad p_k[XY] = p_k[X]p_k[Y]$$

$$(2.9) \quad p_k[X + Y] = p_k[X] + p_k[Y]$$

$$(2.10) \quad p_k[-X] = -p_k[X].$$

This operation is called *plethysm*. Since $\Lambda_{\mathbb{F}}$ is freely generated by the power sums, the plethysm operator can be extended to the other symmetric functions. In fact, using plethysm, one defines the following bases for $\Lambda_{\mathbb{Q}(q)}^n$ and $\Lambda_{\mathbb{Q}(q,t)}^n$, respectively.

Theorem-Definition 1. (*cocharge Hall-Littlewood polynomials*)

There exists a basis $\{\tilde{H}_{\mu}(X; q)\}_{\mu \vdash n}$ for the vector space $\Lambda_{\mathbb{Q}(q)}^n$, which is uniquely characterized by the properties

- (1) $\tilde{H}_{\mu}(X; q) \in \mathbb{Z}[q]\{s_{\lambda} : \lambda \geq \mu\}$,
- (2) $\tilde{H}_{\mu}[(1-q)X; q] \in \mathbb{Z}[q]\{s_{\lambda} : \lambda \geq \mu'\}$,
- (3) $\langle \tilde{H}_{\mu}(X; q), s_{(n)} \rangle = 1$.

Theorem-Definition 2. (*Modified Macdonald polynomials*)

There exists a basis $\{\tilde{H}_{\mu}(X; q, t)\}_{\mu \vdash n}$ for the vector space $\Lambda_{\mathbb{Q}(q,t)}^n$, which is uniquely characterized by the properties

- (1) $\tilde{H}_{\mu}(X; q, t) \in \mathbb{Z}[q]\{s_{\lambda} : \lambda \geq \mu\}$,
- (2) $\tilde{H}_{\mu}[(1-q)X; q, t] \in \mathbb{Z}[q]\{s_{\lambda} : \lambda \geq \mu'\}$,
- (3) $\langle \tilde{H}_{\mu}[X(1-t); q, t], s_{(n)} \rangle = 1$.

It follows from these Theorem-Definitions that

$$(2.11) \quad \tilde{H}_{\mu}(X; 0, t) = \tilde{H}_{\mu}(X; t),$$

$$(2.12) \quad \tilde{H}_{\mu}(X; q, t) = \tilde{H}_{\mu'}(X; t, q).$$

2.0.3. *Kostka-Foulkes and Kostka-Macdonald polynomials.* Let $\tilde{H}_\mu(X; q) = \sum_\lambda \tilde{K}_{\lambda\mu}(q) s_\lambda$, and $\tilde{H}_\mu(X; q, t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q, t) s_\lambda$ be, respectively, the Schur basis expansions of the Hall-Littlewood and Macdonald symmetric functions. The coefficients of the Schur functions are called, respectively, the *cocharge Kostka-Foulkes polynomials*, and the *modified Kostka-Macdonald polynomials*. It is known that $\tilde{K}_{\lambda\mu}(q, t), \tilde{K}_{\lambda\mu}(q) \in \mathbb{N}[q, t]$.

It follows from equation (2.11) and the Schur basis expansions that

$$(2.13) \quad \tilde{K}_{\lambda\mu}(0, t) = \tilde{K}_{\lambda\mu}(t).$$

2.0.4. *Cauchy Identities.* Let $X = \sum x_i$ be an alphabet, and let $\Omega[X] = \exp(\sum_{k=1}^{\infty} p_k(X)/k)$. Then,

$$(2.14) \quad \Omega[X] = \prod_i \frac{1}{1 - x_i} = \sum_{n=0}^{\infty} s_n(X),$$

$$(2.15) \quad \Omega[X] = \prod_i (1 - x_i) = \sum_{n=0}^{\infty} s_{1^n}(X).$$

If $Y = \sum y_i$ is another alphabet, then

$$(2.16) \quad e_n[XY] = \sum_{\mu \vdash n} s_\mu[X] s_{\mu'}[Y],$$

$$(2.17) \quad e_n[XY] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X; q, t] \tilde{H}_\mu[Y; q, t]}{\tilde{h}_\mu(q, t) \tilde{h}'_\mu(q, t)},$$

where $\tilde{h}_\mu(q, t)$ and $\tilde{h}'_\mu(q, t)$ are as in (2.2).

$$(2.18) \quad s_\mu[1 - z] = \begin{cases} (-z)^k (1 - z) & \text{if } \mu = (n - k, 1^k), \\ 0 & \text{otherwise.} \end{cases}$$

2.0.5. *Cauchy's q -binomial theorem.* Let $(z; q)_k = (1 - z)(1 - qz) \cdots (1 - q^{k-1}z)$, and let

$$\begin{bmatrix} k \\ r \end{bmatrix} = \frac{(q; q)_k}{(q; q)_r (q; q)_{k-r}}.$$

Then, the Cauchy q -binomial theorem states that
(2.19)

$$(z; q)_k = \sum_{r=0}^k z^r (-1)^r e_r[1, q, \dots, q^{k-1}] = \sum_{r=0}^k z^r q^{\binom{r}{2}} (-1)^r \begin{bmatrix} k \\ r \end{bmatrix}.$$

3. SYMMETRIC FUNCTIONS $E_{n,k}(X)$.

The family $\{E_{n,k}(X)\}_{k=1}^n$ of symmetric functions are defined by the plethystic identity

$$(3.1) \quad e_n[X \frac{1-z}{1-q}] = \sum_{k=1}^n \frac{(z; q)_k}{(q; q)_k} E_{n,k}(X).$$

Let $0 \leq k \leq r$, and let

$$(3.2) \quad T_{k+1,r} = (-1)^k \sum_{i=0}^k (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix}.$$

Proposition 3.1. For $k = 0, \dots, n-1$,

$$(3.3) \quad s_{(k+1, 1^{n-k-1})}[X/(1-q)] = \sum_{r=k+1}^n T_{k+1,r} \frac{E_{n,r}(X)}{(q; q)_r}.$$

Proof. Using the Cauchy q -binomial theorem, we see that the coefficient of $(-z)^k$ on the right hand side of (3.1) is

$$(3.4) \quad q^{\binom{k}{2}} \sum_{i=0}^{n-k} \begin{bmatrix} k+i \\ k \end{bmatrix} \frac{E_{n,k+i}}{(q; q)_{k+i}}.$$

On the other hand, by the identities (2.16) and (2.18),

$$\begin{aligned}
e_n[X \frac{1-z}{1-q}] &= \sum_{\lambda} s_{\lambda}[\frac{X}{1-q}] s_{\lambda'}[1-z] \\
&= \sum_{\lambda'=(n-r, 1^r)} s_{\lambda}[\frac{X}{1-q}] (-z)^r (1-z) \\
&= \sum_{r=0}^{n-1} s_{(r+1, 1^{n-r-1})}[\frac{X}{1-q}] (-z)^r (1-z) \\
&= s_{1^n}[\frac{X}{1-q}] + (-z)(s_{1^n}[\frac{X}{1-q}] + s_{2, 1^{n-2}}[\frac{X}{1-q}]) + \cdots + (-z)^n s_n[\frac{X}{1-q}].
\end{aligned}$$

Comparing the coefficient of $(-z)^k$ gives, for $k \geq 1$,

$$(3.5) \quad q^{\binom{k}{2}} \sum_{i=0}^{n-k} \begin{bmatrix} k+i \\ k \end{bmatrix}_q \frac{E_{n, k+i}}{(q; q)_{k+i}} = s_{k, 1^{n-k}}[\frac{X}{1-q}] + s_{k+1, 1^{n-k-1}}[\frac{X}{1-q}],$$

and

$$(3.6) \quad \sum_{i=1}^n \frac{E_{n, i}}{(q; q)_i} = s_{1^n}[\frac{X}{1-q}].$$

We take the alternating sums of the equations (3.5) and (3.6) to get

$$s_{k+1, 1^{n-k-1}}[X/(1-q)] = (-1)^k \left(\sum_{j=1}^n \frac{E_{n, j}}{(q; q)_j} \right) + \sum_{i=1}^k (-1)^{k+i} \left(q^{\binom{i}{2}} \sum_{j=0}^{n-i} \begin{bmatrix} i+j \\ i \end{bmatrix} \frac{E_{n, i+j}}{(q; q)_{i+j}} \right).$$

By collecting $E_{n, k}(X)$'s, and using (2.19) we obtain

$$s_{(k+1, 1^{n-k-1})}[X/(1-q)] = \sum_{r=k+1}^n T_{k+1, r} \frac{E_{n, r}(X)}{(q; q)_r}.$$

□

Let S and E be the matrices

$$S = \begin{pmatrix} s_{1^n}[X/(1-q)] \\ s_{2,1^{n-1}}[X/(1-q)] \\ \vdots \\ s_n[X/(1-q)] \end{pmatrix} \text{ and } E = \begin{pmatrix} \frac{E_{n,1}}{(q;q)_1} \\ \frac{E_{n,2}}{(q;q)_2} \\ \vdots \\ \frac{E_{n,n}}{(q;q)_n} \end{pmatrix},$$

respectively, and let T be the transition matrix from E to S , so that $S = TE$. Then, T is an upper triangular matrix with the $k+1, r$ 'th entry

$$T_{k+1,r} = (-1)^k \sum_{i=0}^k (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix}.$$

For example, when $n = 5$,

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & q & (q+1)q & (q^2+q+1)q & (q^3+q^2+q+1)q \\ 0 & 0 & q^3 & q^3(q^2+q+1) & q^3(q^4+q^3+2q^2+q+1) \\ 0 & 0 & 0 & q^6 & q^6(q^3+q^2+q+1) \\ 0 & 0 & 0 & 0 & q^{10} \end{pmatrix}.$$

Then,

$$T^{-1} = \begin{pmatrix} 1 & -q^{-1} & q^{-2} & -q^{-3} & q^{-4} \\ 0 & q^{-1} & -\frac{q+1}{q^3} & \frac{q^2+q+1}{q^5} & -\frac{q^3+q^2+q+1}{q^7} \\ 0 & 0 & q^{-3} & -\frac{q^2+q+1}{q^6} & \frac{q^4+q^3+2q^2+q+1}{q^9} \\ 0 & 0 & 0 & q^{-6} & -\frac{q^3+q^2+q+1}{q^{10}} \\ 0 & 0 & 0 & 0 & q^{-10} \end{pmatrix}.$$

Proposition 3.2. T^{-1} is (necessarily) upper triangular, and its $k+1, r$ 'th entry is equal to

$$(3.7) \quad (T^{-1})_{k+1,r} = (-1)^{r-k} q^{-r(k+1)} T_{k+1,r}.$$

Proof. Let L be the upper triangular matrix with the $k+1, r$ 'th entry

$$L_{k+1,r} = (-1)^{r-k} q^{-r(k+1)} T_{k+1,r} \quad \text{for } r > k.$$

Clearly, TL is an upper triangular matrix, and the $i+1, j$ 'th entry of TL is

$$(3.8) \quad (TL)_{i+1,j} = \sum_{k=1}^n T_{i+1,k} L_{k,j}.$$

It is straightforward to check that $(TL)_{i+1,i+1} = 1$. We use induction on j to prove that for all $i+1 < j$, $(TL)_{i+1,j} = 0$. So, we assume that for all $i+1 < j$, $(TL)_{i+1,j} = 0$, and we are going to prove that for all $i+1 < j+1$, $(TL)_{i+1,j+1} = 0$.

First of all, using the q -binomial identity

$$(3.9) \quad \begin{bmatrix} r \\ m \end{bmatrix} = \begin{bmatrix} r-1 \\ m \end{bmatrix} + \begin{bmatrix} r-1 \\ m-1 \end{bmatrix} q^{r-m}, \text{ for } m \geq 0,$$

it is easy to show that

$$(3.10) \quad T_{i+1,k} = T_{i+1,k-1} + q^i T_{i,k-1}.$$

It follows that

$$(3.11) \quad L_{k+1,j+1} = -q^{-(k+1)} L_{k+1,j} + q^{-k} L_{k,j-1}.$$

Therefore,

$$\begin{aligned} \sum_{k=i+1}^{j+1} T_{i+1,k} L_{k,j+1} &= \sum_{k=i+1}^{j+1} T_{i+1,k} (-q^{-(k+1)} L_{k+1,j} + q^{-k} L_{k,j-1}) \\ &= \sum_{k=i+1}^{j+1} -q^{-k} T_{i+1,k} L_{k,j} + \sum_{k=i+1}^{j+1} q^{-(k-1)} T_{i+1,k} L_{k-1,j}. \end{aligned}$$

Using (3.10) in the last summation, we have

$$\begin{aligned} \sum_{k=i+1}^{j+1} T_{i+1,k} L_{k,j+1} &= \sum_{k=i+1}^{j+1} -q^{-k} T_{i+1,k} L_{k,j} + \sum_{k=i+1}^{j+1} q^{-(k-1)} T_{i+1,k-1} L_{k-1,j} \\ &\quad + \sum_{k=i+1}^{j+1} q T_{i+1,k-1} L_{k-1,j}. \end{aligned}$$

After rearranging the indices, and using the induction hypotheses, the right hand side of the equation simplifies to 0. Therefore, the proof is complete. \square

Corollary 3.3. *Let $A \subseteq \Lambda_{\mathbb{Q}(q)}^n(X)$ be the n -dimensional subspace generated by the set $\{E_{n,k}(X)\}_{k=1}^n$, and let $B \subseteq \Lambda_{\mathbb{Q}(q)}^n(X)$ be the n -dimensional subspace generated by $\{s_{k,1^{n-k}}[\frac{X}{1-q}]\}_{k=1}^n$. Then, $A = B$.*

Proof. It is clear by Proposition 3.2 that $A = B$. The dimension claim follows from Proposition 4.1 below. \square

The expression $(-1)^n p_n = \sum_{k=0}^{n-1} (-1)^k s_{k+1,1^{n-k-1}}$ is the bridge between Schur functions of hook type with the power sum symmetric functions. By the linearity of plethysm we have

$$(-1)^n p_n[X]/(1-q^n) = (-1)^n p_n[X/(1-q)] = \sum_{k=0}^{n-1} (-1)^k s_{k+1,1^{n-k-1}}[X/(1-q)],$$

and therefore

$$(3.12) \quad (-1)^n p_n = (1-q^n) \sum_{k=0}^{n-1} (-1)^k s_{k+1,1^{n-k-1}}[X/(1-q)].$$

Corollary 3.4. *For all $n \geq 1$,*

$$(3.13) \quad (-1)^n p_n = \sum_{r=1}^n \frac{1-q^n}{1-q^r} E_{n,r}.$$

Proof. By Proposition 3.1 and (3.12) we get

$$(3.14) \quad (-1)^n p_n = (1-q^n) \sum_{k=0}^{n-1} \sum_{r=k+1}^n (-1)^k T_{k+1,r} \frac{E_{n,r}(X)}{(q;q)_r}.$$

By rearranging the summations and using the Cauchy's q -binomial theorem once more, we finish the proof. \square

4. HALL-LITTLEWOOD EXPANSION.

Proposition 4.1. For $k = 1, \dots, n$,

$$\frac{E_{n,k}(X)}{(q; q)_k} = \sum_{\mu \in \text{Par}(n,k)} \frac{\tilde{H}_{\mu'}(X; q)}{\tilde{h}_{\mu}(q, 0)\tilde{h}'_{\mu}(q, 0)} = \sum_{\mu \in \text{Par}(n,k)} \frac{\tilde{H}_{\mu'}(X; q)}{(-q)^n q^{2n(\mu')} \prod_{s \in \mu, l_{\mu}(s)=0} (1 - q^{-a_{\mu}(s)-1})}.$$

Proof. Let $Y = (1-t)(1-z)$. Then, by the Cauchy identity (2.17), we have

$$(4.1) \quad \sum_{k=1}^n (z; q)_k \frac{E_{n,k}(X)}{(q; q)_k} = \sum_{\mu \vdash n} \frac{\tilde{H}_{\mu}[X; q, t] \tilde{H}_{\mu}[(1-t)(1-z); q, t]}{\tilde{h}_{\mu}(q, t) \tilde{h}'_{\mu}(q, t)}.$$

The left hand side of the equation (4.1) is independent of the variable t . Since $\tilde{h}_{\mu}(q, 0) \neq 0$, and since $\tilde{h}'_{\mu}(q, 0) \neq 0$, we are allowed to make the substitution $t = 0$ on both sides of the equation.

Note that

$$\tilde{h}_{\mu}(q, 0) \tilde{h}'_{\mu}(q, 0) = \prod_{s \in \mu} q^{a_{\mu}(s)} \prod_{s \in \mu, l_{\mu}(s) \neq 0} (-q^{a_{\mu}(s)+1}) \prod_{s \in \mu, l_{\mu}(s)=0} (1 - q^{a_{\mu}(s)+1})$$

$$(4.3) \quad = (-q)^n \prod_{s \in \mu} q^{2a_{\mu}(s)} \prod_{s \in \mu, l_{\mu}(s)=0} \frac{1 - q^{a_{\mu}(s)+1}}{-q^{a_{\mu}(s)+1}}$$

$$(4.4) \quad = (-q)^n \prod_{s \in \mu} q^{2a_{\mu}(s)} \prod_{s \in \mu, l_{\mu}(s)=0} (1 - q^{-a_{\mu}(s)-1})$$

$$(4.5) \quad = (-q)^n q^{2n(\mu')} \prod_{s \in \mu, l_{\mu}(s)=0} (1 - q^{-a_{\mu}(s)-1}).$$

The equality (4.5) follows from (2.1).

Using the Schur expansion $\tilde{H}_{\mu}(X; q, t) = \sum_{\lambda} \tilde{K}_{\lambda\mu}(q, t) s_{\lambda}$, we see that the plethystic substitution $X \rightarrow (1-z)$, followed by the evaluation at $t = 0$ is the same as the evaluation $\tilde{H}_{\mu}(X, q, 0)$ at $t = 0$, followed by the plethystic substitution $X \rightarrow (1-z)$. Also, by Corollary 3.5.20 of [6], we know that

$$\tilde{H}_{\mu}[1-z; q, t] = \Omega[-zB_{\mu}], \text{ where } B_{\mu} = \sum_{i \geq 1} t^{i-1} \frac{1 - q^{\mu_i}}{1 - q}.$$

Therefore,

$$(4.6) \quad \tilde{H}_\mu[1 - z; q, 0] = \Omega[-zB_\mu]|_{t=0}$$

$$(4.7) \quad = \Omega[-z(1 + q + \cdots + q^{\mu_1-1})]$$

$$(4.8) \quad = \prod_{i=0}^{\mu_1-1} (1 - zq^i)$$

$$(4.9) \quad = (z; q)_{\mu_1}.$$

It follows from (2.11) and (2.12) that

$$(4.10) \quad \tilde{H}_\mu(X; q, 0) = \tilde{H}_{\mu'}(X; q).$$

By combining (4.1), (4.5), (4.9) and (4.10), we get

$$(4.11) \quad \sum_{k=1}^n (z; q)_k \frac{E_{n,k}(X)}{(q; q)_k} = \frac{1}{(-q)^n} \sum_{\mu \vdash n} (z; q)_{\mu_1} \frac{\tilde{H}_{\mu'}(X; q)}{q^{2n(\mu')} \prod_{s \in \mu, l_\mu(s)=0} (1 - q^{-a_\mu(s)-1})}.$$

By comparing the coefficient of $(z; q)_k$ in (4.11), we find that

$$(4.12) \quad \frac{E_{n,k}(X)}{(q; q)_k} = \sum_{\mu \in \text{Par}(n, k)} \frac{\tilde{H}_{\mu'}(X; q)}{(-q)^n q^{2n(\mu')} \prod_{s \in \mu, l_\mu(s)=0} (1 - q^{-a_\mu(s)-1})}.$$

Hence, the proof is complete. \square

Lemma 4.2. *Let $\lambda \vdash n$ be a partition of n . Then,*

$$(4.13) \quad s_\lambda \left[\frac{X}{(1-q)(1-t)} \right] = \sum_{\mu \vdash n} \frac{\tilde{K}_{\lambda'\mu}(q, t) \tilde{H}_\mu(X; q, t)}{\tilde{h}_\mu(q, t) \tilde{h}'_\mu(q, t)}.$$

Proof. This follows from Theorem 1.3 of [3]. \square

Corollary 4.3. *Let $\lambda \vdash n$ be a partition. Then,*

$$(4.14) \quad s_\lambda \left[\frac{X}{1-q} \right] = \sum_{\mu} \frac{\tilde{K}_{\lambda'\mu}(q) \tilde{H}_{\mu'}(X; q)}{\tilde{h}_\mu(q, 0) \tilde{h}'_\mu(q, 0)} = \sum_{\mu} \frac{\tilde{K}_{\lambda'\mu}(q) \tilde{H}_{\mu'}(X; q)}{(-q)^n q^{2n(\mu')} \prod_{s \in \mu, l_\mu(s)=0} (1 - q^{-a_\mu(s)-1})}.$$

Proof. It follows from (2.12) and (2.13) that $\tilde{K}_{\lambda'\mu}(q, 0) = \tilde{K}_{\lambda'\mu'}(0, q) = \tilde{K}_{\lambda'\mu'}(q)$. Since, $\tilde{h}_\mu(q, 0) \tilde{h}'_\mu(q, 0) = (-q)^n q^{2n(\mu')} \prod_{s \in \mu, l_\mu(s)=0} (1 -$

$q^{-a_\mu(s)-1}$), and $\tilde{H}_\mu(X; q, 0) = \tilde{H}_{\mu'}(X; q)$, the proof follows from Lemma 4.2. \square

Theorem 4.4. *Let $1 \leq k \leq r \leq n$, and let $\mu \in \text{Par}(n, r)$. Then,*

$$T_{k,r} = \tilde{K}_{(n-k+1, 1^{k-1})_{\mu'}}(q).$$

Proof. Recall that

$$s_{k+1, 1^{n-k-1}} \left[\frac{X}{1-q} \right] = \sum_{r=k+1}^n T_{k+1,r} \frac{E_{n,r}(X)}{(q; q)_r},$$

where

$$T_{k+1,r} = (-1)^k \sum_{i=0}^k (-1)^i q^{\binom{i}{2}} \begin{bmatrix} r \\ i \end{bmatrix}.$$

Therefore, by Corollary 4.3 and Proposition 4.1 we have

$$\begin{aligned} \sum_{\mu} \frac{\tilde{K}_{(n-k+1, 1^{k-1})_{\mu'}}(q) \tilde{H}_{\mu'}(X; q)}{\tilde{h}_{\mu}(q, 0) \tilde{h}'_{\mu}(q, 0)} &= s_{k, 1^{n-k}} \left[\frac{X}{1-q} \right] \\ &= \sum_{r=k}^n T_{k,r} \sum_{\mu \in \text{Par}(n, r)} \frac{\tilde{H}_{\mu'}(X; q)}{\tilde{h}_{\mu}(q, 0) \tilde{h}'_{\mu}(q, 0)} \\ &= \sum_{\mu \in \bigcup_{r=k}^n \text{Par}(n, r)} \frac{T_{k,r} \tilde{H}_{\mu'}(X; q)}{\tilde{h}_{\mu}(q, 0) \tilde{h}'_{\mu}(q, 0)}. \end{aligned}$$

The theorem follows from comparison of the coefficients of $\tilde{H}_{\mu'}(X; q)$. \square

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